Congruence and Subgroup Lattices

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Congruence relations play a fundamental role in universal algebra. All congruence relations of an algebraic structure form a complete lattice, that has an additional property, namely, every element is a join of compact elements in the lattice. Such lattices are called algebraic. In 1963 Grätzer and Schmidt proved that there is no other general property of congruence lattices: for every algebraic lattice they constructed a suitable algebra with congruence lattice isomorphic to the given lattice. They construct this algebra as a result of an infinite extension procedure. Hence even if the lattice to be represented is finite, the algebra with the given congruence lattice will be infinite. It is a famous open problem in universal algebra **whether every finite lattice can be represented as the congruence lattice of a finite algebra.**

Permutation groups can be considered as multi-unary algebras, where every permutation is regarded as a unary (one-variable) operation. For transitive permutation groups the congruence lattice is the same as the interval in the subgroup lattice consisting of the subgroups containing the stabilizer of a chosen point. In a joint paper with Pavel Pudlák we showed that for a large class of finite lattices the smallest algebra representing such a lattice (if there is any) must be a permutation group. From this it follows that the problem of representing finite lattices as congruence lattices of arbitrary finite algebras is in fact equivalent to a group theoretic one: **Is it true that every finite lattice occurs as an interval in the subgroup lattice of a finite group?**

The finiteness is essential in this problem. Tůma proved that **every algebraic lattice can be represented as an interval in the subgroup lattice of an infinite group.** This way he gave another new proof for the Grätzer–Schmidt Theorem. Here again the constructed group is always infinite, even if the given lattice is finite. Namely, the essential step in the construction uses a free product of groups.

Returning to the problem about intervals in subgroup lattices of finite groups, by imposing conditions on the lattice one can derive restrictions on the structure of the group. One would like to reduce the problem to the case of **almost simple groups**, as it often happens in group theory, and then attack it using the classification of finite simple groups. However, it seems inevitable that another kind of groups, certain **twisted wreath products**, has to be investigated as well. Works of Baddeley, Lucchini, Börner, and Aschbacher yield this dichotomy of reduction of the problem.