Mini-course: Algebraic aspects of Boolean inverse semigroups
Friedrich Wehrung, LMNO UMR 6139, CNRS Université de Caen, 14032 Caen cedex, FRANCE
E-mail address: friedrich.wehrung01@unicaen.fr.

The classical Banach-Tarski paradox states that the unit ball S^2 of the three-dimensional Euclidean space can be decomposed into finitely many pieces which, rearranged under suitable rotations, yield two copies of the original ball. The transformations on S^2 thus involved are only *partial bijections*; in particular they do not form a group under composition. Instead, they form an *inverse semigroup.* Additional special properties satisfied by that inverse semigroup are:

- The idempotents form a generalized Boolean algebra.
- Any two orthogonal elements x and y have a join $x \oplus y$ (here the least common extension of x and y as a partial function).

Such structures have been christened *Boolean inverse semigroups* (say BISs; see [4, 5]). The present mini-course will be based on the author's monograph [8], with borrowings from [9], and with focus on the following aspects:

- The class of all BISs, with *additive* (i.e., preserving orthogonal addition) homomorphisms, is a congruence-permutable variety of algebras (*biases*), sharing aspects of both groups and rings.
- The quotient of any BIS S under Green's relation $\mathscr{D}$ is a partial monoid, whose universal monoid, called the *type monoid* $\operatorname{Typ} S$ of S (cf. [6]), is a *conical refinement monoid.* Type monoids of BISs are exactly the monoids $\mathbb{Z}^+\langle B\rangle /\!\!/ G$ of equidecomposability types of generalized Boolean algebras under group actions; we call such monoids *measurable.*
- Every measurable monoid is a conical refinement monoid. For *countable* monoids the converse holds (this originates in [1, 2]). For cardinalities beyond $\aleph_2$ it fails (cf. [7]).
- The positive cone of any abelian ℓ-group is measurable; the solution can be made *functorial.*
- For any field K and any BIS S, denote by $K\langle S\rangle$ the K-algebra defined by generators S and relations $z = x + y$ (within $K\langle S\rangle$) whenever $z = x \oplus y$ (within S). There is a natural monoid homomorphism $\boldsymbol{f}$ from $\operatorname{Typ} S$ to the nonstable K-theory $\mathrm{V}(K\langle S\rangle)$ of $K\langle S\rangle$. In a few important cases $\boldsymbol{f}$ is an isomorphism. However, in general $\boldsymbol{f}$ may be neither surjective nor one-to-one.
- The classical nonstable K-theory of AF C*-algebras, originating in [3], carries over to type monoids of so-called *locally matricial BISs.*
- Any two BISs S and T have a *tensor product* $S \otimes T$. Moreover, $\operatorname{Typ}(S \otimes T) \cong (\operatorname{Typ} S) \otimes (\operatorname{Typ} T)$.

References

[1] Hans Dobbertin, *On Vaught's criterion for isomorphisms of countable Boolean algebras*, Algebra Universalis **15** (1982), no. 1, 95–114. MR 663956 (83m:06017)

[2] ———, *Refinement monoids, Vaught monoids, and Boolean algebras*, Math. Ann. **265** (1983), no. 4, 473–487. MR 721882 (85e:06016)

[3] George A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44. MR 0397420 (53 #1279)

[4] Mark V. Lawson, *A noncommutative generalization of Stone duality*, J. Aust. Math. Soc. **88** (2010), no. 3, 385–404. MR 2827424 (2012h:20141)

[5] ———, *Non-commutative Stone duality: inverse semigroups, topological groupoids and C^*-algebras*, Internat. J. Algebra Comput. **22** (2012), no. 6, 1250058, 47. MR 2974110

[6] Alistair R. Wallis, *Semigroup and category-theoretic approaches to partial symmetry*, Ph.D. thesis, Heriot-Watt University, Edinburgh, 2013.

[7] Friedrich Wehrung, *Non-measurability properties of interpolation vector spaces*, Israel J. Math. **103** (1998), 177–206. MR 1613568 (99g:06023)

[8] ———, *Refinement Monoids, Equidecomposability Types, and Boolean Inverse Semigroups*, Lecture Notes in Mathematics, vol. 2188, Springer, Cham, 2017. MR 3700423

[9] ———, *Varieties of Boolean inverse semigroups*, J. Algebra **511** (2018), 114–147. MR 3834766